APPROXIMATE SOLUTION OF CAUCHY'S PROBLEM FOR LAPLACE'S EQUATION APPLICABLE TO THE PROBLEM OF SHAPING OF DENSE SPATIALLY INHOMOGENEOUS BEAMS OF CHARGED PARTICLES

PMM Vol. 35, No. 4, pp. 656-668
V. N. DANILOV and V. A. SYROVOI
(Moscow)
(Received 30 November, 1970)

A solution of the problem of shaping (Sect. 1) of a number of three-dimensional beams is given in the form of asymptotic expansions. The results are compared with exact expressions which determine the shaping electrodes for a plane flow along circular trajectories (Sect. 2). From the paraxial approximation for the electrostatic beams, cases which satisfy the conditions of a full spatial charge on the emitter, without disturbing the regularity of this approximation (Sect. 3) are selected. Quasi-axially symmetric beams (Sect. 4) and a quasi-cylindrical domain of arbitrary section (Sect. 5) are considered.

The hydrodynamic theory of intense beams of charged particles represents one of the branches of the mechanics of continuous medium. However, the asymptotic methods, although used widely and for a long time in other branches of mechanics, began to find application in this field only recently. The inverse problem or the problem of synthesis which appears when a system with desired charachteristics is constructed, consists of two parts: the internal problem, which deals with solutions of the equations of the beam, and the external problem, connected with determining the shaping electrodes which provide the realization of the computed flow. The Cauchy problem for the Laplace equation represents the mathematical expression for the latter. For the solution of the internal problem for narrow beams the asymptotic method of the extension or the narrow strip type [4, 5] have been successfully used in [1-3], although the study of the problem of shaping was complicated by the existence of singularities at the flow boundary. An approximate solution of this problem with singularities present in the Cauchy conditions based on the multiscale and factorization method, is given below.

1. Formulation of the problem. In the system x^i $(i = 1, 2, 3)$ with a metric tensor g_{ik}, $g = \det g_{ik}$ the Laplace equation has the following form:

$$\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ik}\frac{\partial\varphi}{\partial x^k}\right) = 0 \qquad (i,k = 1, 2,3) \tag{1.1}$$

Here the potential and its normal derivative on the surface Σ, separating the region Ω, occupied by the charges from the region free from the charges are both assumed known

$$\varphi|_\Sigma = V(P), \qquad \partial\varphi/\partial n|_\Sigma = F(P), \qquad P \in \Sigma \tag{1.2}$$

For the complex, spatially heterogeneous beams of sufficient complexity, the Cauchy conditions can be obtained in two ways: (1) as few exact solutions of the equations of a beam in Ω, or (2) the paraxial approximation solutions. The geometrical complexity of the problems under discussion makes it practically impossible to solve them exactly. Therefore this study will be conducted within the limits of asymptotic approach. We shall construct a solution in a narrow strip near a sufficiently smooth axial curve $\mathbf{r} = \mathbf{R}(l)$, characterized by its curvature $k(l)$ and its torsion $\varkappa(l)$.

Estimation of the order of the expansion terms is aided by introducing an order-of-smallness index μ, which is inserted wherever a small parameter μ_*, appears on changing to dimensionless quantities. The latter parameter represents the ratio of the width of the strip a_* to the charachteristic longitudinal dimension L_*. In the course of solving the external problem both the curvature and the torsion of the axial curve are assumed to be the order of μ i.e. μk, and $\mu\varkappa$. In solving the beam equations, the paraxial approach [3] consists of expanding the functions sought into power series in the small parameter ε_*, the latter charachterizing the narrowness of the region Ω and its error is of the order of ε_*^3. In every real problem it is possible to establish a relation between ε and μ, e.g. $\mu = \varepsilon^{1/2}$, which permits, bearing in mind the error of the solution in Ω, to determine the number of terms in the expansion for the external problem

$$\varphi = \sum_{n=0} \varphi\langle n\rangle \mu^n \tag{1.3}$$

which will provide the prescribed accuracy. Passage to the dimensionless coordinates (extension) is not essential and therefore instead of ε_*, and μ_* we can use symbols ε and μ provided that in the final formulas they are taken as equal to unity.

It is known that the investigation of flows issuing from the emitting surfaces, the Cauchy conditions on Σ are irregular functions if the initial velocity is zero. Several singularities may appear. The simplest example is a branch singularity ($\varphi \sim l^{4/3}$) which corresponds to an electrostatic flow under thecondition of full spatial charge. In the following we shall be concerned mainly with such singularities.

Quasi-one-dimensional asymptotic expansions of type (1.3) shown in [3] or the closely related series in powers of the coordinate normal to Σ [6, 7] become nonuniform on approaching the singularity. To construct expansions (1.3) which are equally usable over the entire strip, we must select the singularities correctly. This can be done by introducing an additional variable similar to the coordinate intended for describing the basically two-dimensional distribution of potential near a singularity (multiscale method [5]). Thus, l is replaced by two longitudinal coordinates z and L, and the index μ precedes the derivatives with respect to L

$$l \to z, L, \frac{\partial}{\partial l} \to \frac{\partial}{\partial z} + \mu\frac{\partial}{\partial L}, \quad \frac{\partial^2}{\partial l^2} \to \frac{\partial^2}{\partial z^2} + 2\mu\frac{\partial^2}{\partial z\partial L} + \mu^2\frac{\partial^2}{\partial L^2} \tag{1.4}$$

As L describes smooth and slowly changing functions, it is convenient to call it the "slow" coordinate in contrast to the "fast" variable z. In the final formulas we must set $l = z = L$.

2. The shaping of a plane flow with circular trajectories. The equations of beam for this case were solved in [8]. The problem is that of solving the Laplace equation.

$$\partial^2\varphi/\partial\sigma^2 + \partial^2\varphi/\partial\psi^2 = 0, \quad \sigma = \ln R \tag{2.1}$$

which satisfies the following conditions when $\sigma = 0$:

$$\varphi\,|_{\sigma=0} = \left(\sin\frac{3}{2}\psi\right)^{4/3} = V(\psi), \qquad \frac{\partial\varphi}{\partial R}\bigg|_{\sigma=0} = -2V(\psi) = F(\psi) \tag{2.2}$$

Here R, ψ are the polar coordinates. The problem (2.1), (2.2) has the following exact solution:

$$\varphi = \operatorname{Re} V(w) + \operatorname{Im}\int_0^w F(t)\,dt, \qquad w = \psi + i\sigma \tag{2.3}$$

The expansions given in [9] for the integral appearing in (2.3) become unsuitable near the singularities($\psi = 0, 120°$)of the functions (2.2), while [10] gives the equipotential surfaces constructed by integrating the ordinary differential equation of the equipotential To obtain the singularity in its simplest form, we supplement ψ with another angular variable θ and factorize the conditions (2.2)

$$V = \left(\sin\frac{3}{2}\psi\right)^{4/3} = \left(\frac{3}{2}\theta\right)^{4/3}, A(\Psi),\ F = -2V,\ A(\Psi) = \left(\frac{\sin {}^3/_2\Psi}{{}^3/_2\Psi}\right)^{4/3} \tag{2.4}$$

The variable θ is identical to ψ and is only used to describe the singularity. The factorization yields the irregular part in its simplest form, and the function A (Ψ) is regular for $0 \leqslant \psi \leqslant 60°$, i.e. in the region to which we confine our investigations by virtue of the symmetry of the Cauchy conditions with respect to the ray $\psi = 60°$. Equation (2.1) then becomes

$$\frac{\partial^2\varphi}{\partial\sigma^2} + \frac{\partial^2\varphi}{\partial\theta^2} = -2\mu\,\frac{\partial^2\varphi}{\partial\theta\,\partial\Psi} - \mu^2\,\frac{\partial^2\varphi}{\partial\Psi^2} \tag{2.5}$$

Writing the solution in the form (1.3) we obtain the following expressions for $\varphi\langle n\rangle$:

$$\varphi\langle n\rangle = \left(-\frac{1}{2}\right)^n A^{(n)}(\Psi)\,\Phi_n(w, \bar{w}), \quad \Phi_n(w, \bar{w}) = \int_{\bar{w}}^{w} dx \int_x^w F_n(x, y)\,dy$$

$$F_n = \Phi_{n-1,\,w} + \Phi_{n-1,\,\bar{w}} - \Phi_{n-2}, \quad w = \theta + i\sigma, \quad \bar{w} = \theta - i\sigma, \quad \Phi_{n,\,w} = \partial\Phi_n / \partial w$$

$$\Phi_0 = \frac{1}{2}\left(w^{4/3} + \bar{w}^{4/3}\right) + \frac{1}{2i}\cdot\frac{6}{7}\left(w^{7/3} - \bar{w}^{7/3}\right) \tag{2.6}$$

The formulas (2.6) yielded seven approximations, including $\varphi\ \langle 7\rangle$. Figure 1 gives some idea about the region within which the knowledge of the potential leads to a complete solution of the problem of shaping; the region is bounded by the rays $\psi = 0°$ and $\psi = 60°$ and the zero equipotential when $R \geqslant 1$. For $\varphi \sim 1$ the relative error was found to be $\delta_n = |1 - \varphi_n / \varphi_{ex}|\%$, where φ_n is the potential of the n-th approximation and φ_{ex} its exact value. At the points farthest away from the boundary, the position of the zero equipotential is used to estimate the accuracy. It should be noted that although the singularity at the coordinate origin was not taken into account in the expansion (2.6), the expansion should be regarded as sufficiently satisfactory. Using $\varphi_{\langle 7\rangle}$ it was found possible to compute the potential in the region $0.1 \leqslant R \leqslant 1$ with an error not exceeding 1% and to obtain practically exact values for $R \geqslant 1$. In the strip $0.6 \leqslant R \leqslant 1.4$, φ_2 gives a solution with an error of less than 2% and φ_3 less than 0.5%. For $\varphi_2 \varphi_3$ the errors in the values of the zero equipotential coordinates are:

$\psi = 60°$, $\delta_2 = 11.6\%$, $\delta_3 = 8.8\%$; $\psi = 40°$, $\delta_2 = 8.8\%$, $\delta_3 = 6.6\%$; $\psi = 20°$, $\delta_2 = 1.7\%$, $\delta_3 = 0.07\%$.

It can be expected that the approximate approach applied to physical problems of the same type, will produce errors of the same order for different rpoblems within the same class.

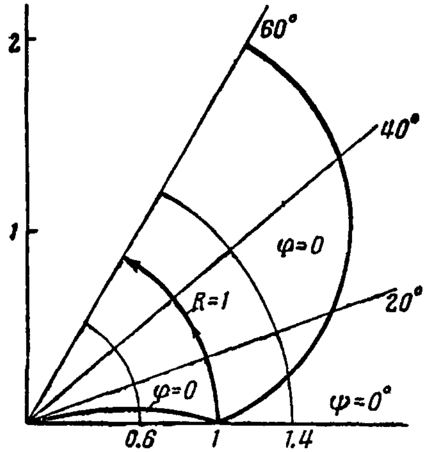

Fig. 1.

3. Study of the paraxial equations of the beam and selection of the singularity. We have noted above that the solutions, determining spatially inhomogeneous flows of sufficient geometrical complexity can be obtained, almost exclusively, within the framework of the paraxial approximation; the construction of this approximation is given in [3] for axial beams. Here we shall only consider the solutions for an electrostatic flow, corresponding to almost homogeneous transverse density distribution. The approximation is formulated in the coordinates l, s, q, connected with the rectangular coordinates by

$$\mathbf{r} = \mathbf{R}(l) + \mathbf{s}(l)s + \mathbf{q}(l)q \tag{3.1}$$

where l is the arc length of the axial curve, $\mathbf{s}$, and $\mathbf{q}$ are the normal and binormal unit vectors. The trajectories $\xi = \text{const}$, $\eta = \text{const}$ are given by the formulas

$$s = \alpha(l)\xi + \beta(l)\eta, \qquad q = \mu(l)\xi + \nu(l)\eta \tag{3.2}$$

where ξ, η are the initial values of transverse coordinates on the surface of the emitter. The expansion of the potential in the domain filled with charges corresponding to this case has the form

$$\varphi = {}^1/_2\varepsilon^{-2}V^2(l) + \varepsilon^{-1}kV^2s + {}^1/_2V^2(3k^2s^2 - \varkappa^2s^2 - \varkappa^2q^2) + {}^1/_2\Psi_{ss}s^2 + \Psi_{sq}sq + {}^1/_2\Psi_{qq}q^2, \quad D = \alpha\nu - \beta\mu$$
$$D\Psi_{ss} = \alpha^{\cdot\cdot}\nu - \beta^{\cdot\cdot}\mu - 2\varkappa V(\mu^{\cdot}\nu - \nu^{\cdot}\mu), \quad D\Psi_{sq} = \beta^{\cdot\cdot}\alpha - \alpha^{\cdot\cdot}\beta - \varkappa V(\nu^{\cdot}\alpha - \alpha^{\cdot}\nu + \beta^{\cdot}\mu - \mu^{\cdot}\beta)$$
$$D\Psi_{qq} = \nu^{\cdot\cdot}\alpha - \mu^{\cdot\cdot}\beta - 2\varkappa V(\alpha^{\cdot}\beta - \beta^{\cdot}\alpha), \quad \alpha^{\cdot} = Vd\alpha/dl \tag{3.3}$$

In these formulas $k = k(l)$, $\varkappa = \varkappa(l)$ is the curvature and the torsion of the axis, $V = V(l)$ is the axial longitudinal velocity, ε is the index of smallness governing the relative width of the beam and functions α, β, μ, ν, k, $\varkappa$, V are connected by the relations

$$\mu'\nu - \nu'\mu + \alpha'\beta - \beta'\alpha = -2\varkappa D, \qquad V' = dV/dl$$
$$D^{\cdot\cdot} - 2(\alpha^{\cdot}\nu^{\cdot} - \beta^{\cdot}\mu^{\cdot}) + [2(k^2 + \varkappa^2)V^2 + (VV')']D = J/V, \quad J = \text{const} \tag{3.4}$$

Within the limits of approximation of (3.3), an emission with zero velocity can take place only from a flat source. In [1] the rectangular coordinates ξ and ζ, related to the trajectories $\xi = \text{const}$ are used to construct the paraxial approximation for an axisymmetric flow with a rectilinear axis. The conditions of thermal emission can be satisfied on the curved surface $\zeta = 0$. We note that the method of deformed coordinates [5] implies the necessity of constructing the asymptotic expansions for the problems with singularities situated at the surfaces in terms of the curvilinear coordinates related to the charachteristics (in this case to the trajectories). Thus we extend the analysis to include the curvilinear emitter by replacing the coordinates $l,\ s,\ q$ with the coordinates $\zeta,\ \xi,\ \eta$

$$\zeta = l + {}^1/_2\varepsilon^{-2}(Q_{ss}s^2 + 2Q_{sq}sq + Q_{qq}q^2), \qquad \alpha' = d\alpha / dl \qquad (3.5)$$
$$DQ_{ss} = \alpha'\nu - \beta'\mu, \quad DQ_{sq} = \beta'\alpha - \alpha'\beta, \quad DQ_{qq} = \nu'\alpha - \mu'\beta$$

Deformation of the coordinate l in accordance with (3.5) and with the accuracy of up to ε^3 ensures the selection of the singularity in the expansion (3.3).

$$\varphi = {}^1/_2V^2\,\{\varepsilon^{-2} + 2\varepsilon^{-1}ks + (3k^2s^2 - \varkappa^2s^2 - \varkappa^2q^2) + D^{-1}\,[\alpha''\nu - \beta''\mu - 2\varkappa\,(\mu'\nu - \nu'\mu)]\,s^2 + 2D^{-1}\,[\beta''\alpha - \alpha''\beta - \varkappa\,(\nu'\alpha - \alpha'\nu + \beta'\mu - \mu'\beta)] \times sq + D^{-1}\,[\nu''\alpha - \mu''\beta - 2\varkappa\,(\alpha'\beta - \beta'\alpha)]\,q^2\} \qquad (3.6)$$

Since the longitudinal coordinate is only weakly deformed, it can be assumed that all the functions of l, except V, remain, after the passage from $l,\ s,\ q$ to $\zeta,\ \xi,\ \eta$ the same functions of the new longitudinal coordinate ζ. For s and q in (3.6) we should use formulas (3.2). We note that for $\varkappa \neq 0$ the system $\zeta,\ \xi,\ \eta$ is no longer orthogonal.

If the emitting part of the start surface is bounded by a closed contour $\xi = \xi\,(t)$ and $\eta = \eta\,(t)$, then the boundary of the flow is defined by the following parametric equations:

$$s = \alpha(\zeta)\,\xi\,(t) + \beta\,(\zeta)\,\eta\,(t), \qquad q = \mu\,(\zeta)\xi\,(t) + \nu\,(\zeta)\,\eta\,(t) \qquad (3.7)$$

Expansion (3.6) is regular over the whole region including the emitter, provided that V is the only irregular function and that the determinant D does not approach zero anywhere. Let us assume that these conditions are fulfilled; then, with emission limited by a spatial charge, we obtain the following expressions for potential and the field at the boundary and for the axial velocity:

$$\varphi = \zeta^{4/3}\,\Phi\,(\zeta,\ t), \qquad \partial\varphi / \partial s = \zeta^{1/3}\,E\,(\zeta,\ t), \qquad V = \zeta^{2/3}\,W\,(\zeta) \qquad (3.8)$$

Let the functions $\varkappa,\ \alpha,\ \beta,\ \mu,\ \nu$, which define the shape of the beam be chosen so that the first relation of (3.4) is satisfied. Then V can be found by integration of the second equation of (3.4). Substituting (3.8) in (3.4) and replacing l by ζ, we find

$$D\zeta^2W\,(WW'' + W'^2) + ({}^8/_3D\zeta + D'\zeta^2)\,W^2W' + [{}^2/_9D + 2\,(k^2 + \varkappa^2)\,D\zeta^2 - 2\,(\alpha'\nu' - \beta'\mu')\,\zeta^2 + {}^2/_3D'\zeta + D''\zeta^2]\,W^3 = J, \quad W' = dW / d\zeta \qquad (3.9)$$

For $\zeta = 0$ we have ($\alpha = \nu = 1,\ \beta = \mu = 0,\ D = 1$) and the following conditions necessary for the integration of (3.9)

$$\zeta = 0, \quad W = \left(\frac{9J}{2}\right)^{1/3}, \qquad W' = -\frac{4}{15}\left(\frac{9J}{2}\right)^{1/3} D' \qquad (3.10)$$

The value of the derivative W' given in (3.10) ensures that the function is regular and follows from the expansions given in [11]; J denotes the density of the emission flow. Within the limits of the considered approximation, this density of flow must be homogeneous. The start surface is defined by the equation $\zeta = 0$ and, as we can see from (3.5) it represents a paraboloid approximating any sufficiently smooth surface with the accuracy of up to ε^3.

The numerical integration of (3.9) does not present any difficulty and can be performed using a uniform interval commensurable with the width of the beam. Seven functions α, β, μ, ν, k, $\varkappa$, V are related by only two relations (3.4). This makes the paraxial approximation very flexible, especially for constructing three-dimensional flows.

4. Shaping of quasi-axisymmetric beams. Quasi-cylindrical coordinates l, ρ, ϑ are connected with l, s, q given in (3.1), by the relations

$$s = \rho \cos \vartheta, \qquad q = \rho \sin \vartheta$$

The metric on l, ρ, ϑ is given by

$$dr^2 = [(1 - k\rho \cos \vartheta)^2 + \varkappa^2\rho^2]\, dl^2 + d\rho^2 + \rho^2 d\vartheta^2 + 2\varkappa\rho^2 dl d\vartheta \tag{4.1}$$

The determinant of the metric tensor is

$$g = |g_{ik}| = (1 - k\rho \cos \vartheta)^2 \rho^2$$

The Laplace equation in quasi-cylindrical coordinates has the form

$$\frac{\partial^2\varphi}{\partial l^2} - 2\varkappa \frac{\partial^2\varphi}{\partial l \partial\vartheta} + \frac{1}{\rho}(1 - 3k\rho\cos\vartheta + 2k^2\rho^2\cos^2\vartheta)\frac{\partial\varphi}{\partial\rho} +$$
$$+ (1 - k\rho\cos\vartheta)^2\frac{\partial^2\varphi}{\partial\rho^2} + \frac{\varkappa k\rho\sin\vartheta}{1 - k\rho\cos\vartheta}\frac{\partial\varphi}{\partial l} + \frac{k\sin\vartheta}{\rho}\Big(1 - k\rho\cos\vartheta -$$
$$- \frac{\varkappa^2\rho^2}{1 - k\rho\cos\vartheta}\Big)\frac{\partial\varphi}{\partial\vartheta} + \frac{1}{\rho^2}[(1 - k\rho\cos\vartheta)^2 + \varkappa^2\rho^2]\frac{\partial^2\varphi}{\partial\vartheta^2} = 0 \tag{4.2}$$

To obtain the singularity we must use the relations (1.4). The smoothness of the axial curve means that $k = k(L)$, $\varkappa = \varkappa(L)$. We seek the solution in the form

$$\varphi = \sum_{n=0} \varphi\langle n\rangle \mu^n, \qquad \varphi\langle n\rangle = \sum_{p=0} (\Phi_{np}\cos p\vartheta + \Psi_{np}\sin p\vartheta) \tag{4.3}$$

Then the functions Φ_{np} and Ψ_{np} will satisfy the equation

$$\frac{\partial^2 S_{np}}{\partial z^2} + \frac{1}{\rho}\frac{\partial S_{np}}{\partial\rho} + \frac{\partial^2 S_{np}}{\partial\rho^2} - \frac{p^2}{\rho^2}S_{np} = T_{np} \tag{4.4}$$

For a plane axial curve, we have $\varkappa = 0$ and

$$T_{np} = k\Big\{\Big(\frac{3}{2}\frac{\partial}{\partial\rho} + \rho\frac{\partial^2}{\partial\rho^2}\Big)[S_{n-1,\,p+1} + (1 + \delta_{ip}{}^{\Phi})S_{n-1,\,p-1}] -$$
$$- \frac{1}{\rho}\Big(p + \frac{1}{2}\Big)(p + 1)S_{n-1,\,p+1} - \frac{1}{\rho}\Big(p - \frac{1}{2}\Big)(p - 1)S_{n-1,\,p-1}\Big\} +$$
$$+ k^2\Big\{\Big(-\frac{1}{2}\rho\frac{\partial}{\partial\rho} - \frac{1}{4}\rho^2\frac{\partial^2}{\partial\rho^2}\Big)[S_{n-2,\,p+2} + (2 + \delta_{1p}{}^{\Phi} - \delta_{1p}{}^{\Psi})S_{n-2,\,p} +$$
$$+ (1 + \delta_{2p}{}^{\Phi})S_{n-2,\,p-2}] + \frac{1}{4}(p + 2)(p + 1)S_{n-2,\,p+2} + \frac{1}{2}p^2 S_{n-2,\,p} +$$
$$+ \frac{1}{4}(p - 2)(p - 1)S_{n-2,\,p-2}\Big\} - 2\frac{\partial^2}{\partial z\partial L}S_{n-1,\,p} - \frac{\partial^2}{\partial L^2}S_{n-2,\,p}$$
$$\delta_{ip}{}^{\Phi} = 1, \quad i = p, \quad S_{np} = \Phi_{np}; \quad \delta_{ip}{}^{\Psi} = 1, \quad i = p, \quad S_{np} = \Psi_{np} \tag{4.5}$$

When $\varkappa \neq 0$ the right hand sides of (4.4) assume a more complex form, as the functions Φ_{np}, Ψ_{np} cease to be independent. Additional terms in T_{1p}, T_{2p} are given below without separating the different scales in l

$$\Delta T_{1p} = \pm 2\varkappa p\frac{\partial}{\partial l}S_{0p}$$

$$\Delta T_{2p} = \pm 2\varkappa p \frac{\partial}{\partial l} S_{1p} + \varkappa^2 p^2 S_{0p} + \frac{1}{2} \varkappa k p \frac{\partial}{\partial l} [\mp S_{0,\, p+1} \pm (1 + \delta_{1p}{}^{\Phi}) S_{0,\, p-1}]$$

The upper sign corresponds to the case $S_{np} = \Psi_{np}$, and the lower sign to $S_{np} = \Phi_{np}$. The solution of Eq. (4.4), when the Cauchy conditions are given on the boundary

$$\rho = \rho_e(t), \qquad l = l_e(t) \tag{4.6}$$

is done by the Riemann method. The Riemann function $G(\lambda)$ is a hypergeometric function [11], which can also be expressed in terms of a Legendre function

$$G = F(^1/_2 + p,\ ^1/_2 - p;\ 1;\ \lambda) = P_{p-^1/_2}(\nu)$$

$$\lambda = -\frac{(\rho - \rho_c)^2 + (l - l_c)^2}{4\rho\rho_c}, \qquad \nu = 1 - 2\lambda = \frac{\rho^2 + \rho_c^2 + (l - l_c)^2}{2\rho\rho_c} \tag{4.7}$$

Here ρ_c, l_c are coordinates of the observation point at which the value of S_{np} is computed. The symbols given here are explained in greater detail in [11]. If S_0, S_{v0} are the values of the function and its normal derivative at the boundary $v = 0$

$$l + i\rho = l_e(w) + i\rho_e(w), \qquad w = u + iv$$
$$S|_{v=0} = S_0(t) \qquad \partial S / \partial v|_{v=0} = S_{v0}(t) \tag{4.8}$$

the solution of problem (4.4), (4.8) is given by

$$S = \mathrm{Re}\left\{\left[\frac{\rho_e(w)}{\rho}\right]^{1/2} S_0(w) + \int_0^v \left[\left(\frac{\rho_e}{\rho}\right)^{1/2}\left(S_{v0} + \frac{\beta}{2\rho_e} S_0\right) F\left(\frac{1}{2} + p, \frac{1}{2} - p, 1; \lambda_e\right) - \right.\right.$$
$$- \left(\frac{1}{4} - p^2\right)\frac{S_0}{2\rho_e{}^{1/2}\rho^{3/2}}\left(\frac{\rho^2 - \rho_e^2 + [l_e - l]^2}{2\rho_e}\beta + [l_e - l]\alpha\right) F\left(\frac{3}{2} + p, \frac{3}{2} - \right.$$
$$\left.\left. - p, 2; \lambda_e\right)\right] d\xi - \frac{1}{\rho^{1/2}} \int_0^v d\xi \int_0^{v-\xi} K(u + i\xi, \eta)\, d\eta \Bigg\}, \quad K(u, v) = \rho^{1/2} G T, \qquad \zeta = u + i\xi \tag{4.9}$$

$$\lambda = -\frac{[\rho(\zeta, \eta) - \rho(u, v)]^2 + [l(\zeta, \eta) - l(u, v)]^2}{4\rho(\zeta, \eta)\rho(u, v)}, \qquad \lambda_e = -\frac{(\rho_e - \rho)^2 + (l_e - l)^2}{4\rho_e\rho}$$

The functions S_0, S_{v0}, ρ_e, l_e, $\alpha = d\rho_e / dt$, $\beta = dl_e / dt$ have $\zeta = u + i\xi$; $\rho = \rho(u, v)$, $l = l(u, v)$. as their argument. Formula (4.9) will be used as a basis for further specialization and simplification.

It can be shown that for electrostatic flows with emission bounded by a spatial charge, the flow boundary is a regular curve. However, in more complex cases (magnetic field; temperature-limited emission etc.) both the boundary and the Cauchy conditions will be determined explicitly in terms of functions with singularities the charachter of which will not be fully described by the simplest power relationships given in Sect. 3.

A noticeable simplification of the formula (4.9) can be achieved by considering the electrostatic beams with regular boundary regarded as a quasi-cylinder, the radius of which changes only on account of the slow coordinate L.

4.1. The quasi-cylindrical boundary is given by the relations

$$\rho = \rho_e(t) = \rho_0(L), \qquad z = z_e(t) = t \tag{4.10}$$

In this case

$$z + i\rho = w + i\rho_0(L), \quad \alpha(t) = 0, \quad \beta(t) = 1$$
$$\rho = v + \rho_0(L), \quad z = u, \quad w = u + iv = z + i(\rho - \rho_0) \tag{4.11}$$

Taking into account (4.10) and (4.11) we obtain in place of (4.9)

$$S = \left[\frac{\rho_0(L)}{\rho}\right]^{1/2} \operatorname{Re} S_0(w) + \int_{\rho_0}^{\rho} \left\{\left(\frac{\rho_0}{\rho}\right)^{1/2} F\left(\frac{1}{2} + p, \frac{1}{2} - p, 1; \lambda_e\right) \times \right.$$
$$\times \operatorname{Re}\left[S_{v0}(\zeta) + \frac{1}{2\rho_0} S_0(\zeta)\right] - \left(\frac{1}{4} - p^2\right) \frac{\rho^2 - 2\rho_0^2 + 2\rho_0\sigma - \sigma^2}{2(\rho_0\rho)^{3/2}} \times$$
$$\left. \times F\left(\frac{3}{2} + p, \frac{3}{2} - p, 2; \lambda_e\right) \operatorname{Re} S_0(\zeta)\right\} d\sigma - \frac{1}{\rho^{1/2}} \int_{\rho_0}^{\rho} d\sigma \int_0^{\rho} r^{1/2} \times$$
$$\times F\left(\frac{1}{2} + p, \frac{1}{2} - p, 1; \lambda\right) \operatorname{Re} T(\zeta, r)\, dr, \quad T(\zeta, r) = T(z, \rho)\Big|_{z \to \zeta,\ \rho \to r} \tag{4.12}$$
$$\lambda_e = \frac{(\sigma - \rho)(\sigma + \rho - 2\rho_0)}{4\rho\rho_0}, \quad \lambda = \frac{(\sigma - \rho_0)^2 - (r - \rho)^2}{4r\rho} \quad \zeta = z + i(\sigma - \rho_0)$$

It is clear that arguments λ and λ_e have become real and that $\rho_0(L)$, which is responsible for the geometry of the flow, is not affected by the differential operators with respect to the fast coordinates. When the boundary is a real cylinder, the expression for the solution in quasi-cylindrical coordinates l, ρ, ϑ is obtained from (4.12) when $\rho_0(L) = 1$.

4.2. Quasi-conical boundary in quasi-cylindrical coordinates. A conical boundary in the l, ρ, ϑ coordinates is the simplest example of the surface with a singularity, for which the relation (4.10) and the solution (4.12) cannot apply. It is clear that the introduction of a supplementary variable z will enable us to consider e.g. the flows the boundary of which is almost conical near the coordinate origin and changes into a quasi-cylinder at some distance from it

$$\rho = z(1 + L)^{-1} \rho_0(L), \qquad \rho_0(0) = \text{const}$$

Let us introduce the quasi-spherical coordinates r, θ

$$r^2 = z^2 + s^2 + q^2, \qquad \theta = \operatorname{arc\,tg}(\rho / z) \tag{4.13}$$

We define the quasi-cone in z, L, ρ, ϑ by the following parametric expressions:

$$\rho = \rho_e(t) = e^t \sin\theta_0 R(L) = \alpha(t), \qquad z = z_e(t) = e^t \cos\theta_0 = \beta(t) \tag{4.14}$$

Setting $R(0) = 1$, we find that θ_0 has a meaning of the angle of the cone, tangential to the quasi-cone (4.14) at the coordinate origin. Substitution of (4.14) into (4.9) leads to the following result:

$$S = \left[\frac{R\sin\theta_0}{A(v, L)}\right]^{1/2} \operatorname{Re}\left[e^{iv/2} S_0(w)\right] + \int_0^v \left\{\left[\frac{R\sin\theta_0}{A(v, L)}\right]^{1/2} \operatorname{Re}\left[e^{i\xi/2}\left\{S_{v0}(\zeta) + \frac{\operatorname{ctg}\theta_0}{2R} S_0(\zeta)\right\}\right] \times \right.$$
$$\times F\left(\frac{1}{2} + p, \frac{1}{2} - p, 1; \lambda_e\right) - \left(\frac{1}{4} - p^2\right) C(L) \frac{\cos\theta_0 \cos\xi - B(v, L)}{2[R\sin\theta_0 A(v, L)]^{3/2}} \operatorname{Re}\left[e^{i\xi/2} S_0(\zeta)\right] \times$$
$$\left. \times F\left(\frac{3}{2} + p, \frac{3}{2} - p, 2; \lambda_e\right)\right\} d\xi - \int_0^v d\xi \int_0^{v-\xi} \left[\frac{A(\eta, L)}{A(v, L)}\right]^{1/2} \operatorname{Re}\left[e^{i\xi/2} T(\zeta, \eta)\right] \times$$
$$\times F\left(\frac{1}{2} + p, \frac{1}{2} - p, 1; \lambda\right) d\eta$$

$$\lambda_e = \frac{C(L)(\cos v - \cos \xi)}{2R \sin \theta_0 A(v, L)}, \quad \lambda = \frac{C(L)[\cos(v - \eta) - \cos \xi]}{2A(\eta, L)A(v, L)}, \quad w = u + iv, \ \zeta = u + i\xi$$
$$u = \ln[r / C(L)], \quad v = \operatorname{arc\,tg}[(\operatorname{tg}\theta - R\operatorname{tg}\theta_0)/(1 + R\operatorname{tg}\theta_0 \operatorname{tg}\theta)]$$
$$T(\zeta, \eta) = T(u, v)_{u \to \zeta,\ v \to \eta}, \quad A(x, L) = \sin x \cos\theta_0 + R\cos x \sin\theta_0$$
$$B(x, L) = \cos x \cos\theta_0 - R\sin x \sin\theta_0, \quad C(L) = \cos^2\theta_0 + R^2\sin^2\theta_0 \qquad (4.15)$$

We see that the expression (4.15) is not more complicated than (4.12) and that λ_e and λ are again real. When $R(L) = 1$, we obtain a proper cone in quasi-cylindrical coordinates and further simplification takes place.

Until now, the Cauchy conditions for the functions S_{np} were assumed known. The value of the potential and of its derivative at boundary Σ of the beam can be obtained from the solution of the inner problem (see Sect. 4 for the quasi-cylinder) and may contain the small parameter ε, specifying the narrowness of the domain occupied by the charges. Writing these functions in the form of series similar to (4.3), we obtain

$$\varphi|_\Sigma = V(t, \vartheta, L, \varepsilon) = \sum_{n=0} V_n^\circ(t, \vartheta, L)\varepsilon^n = \sum_{n=0} V_n(t, \vartheta, L)\mu^n$$

$$\partial\varphi / \partial v|_\Sigma = F(t, \vartheta, L, \varepsilon) = \sum_{n=0} F_n^\circ(t, \vartheta, L)\varepsilon^n = \sum_{n=0} F_n(t, \vartheta, L)\mu^n \qquad (4.16)$$

$$V_n(t, \vartheta, L) = \sum_{p=0} (\vartheta_{np}^c \cos p\vartheta + \vartheta_{np}^s \sin p\vartheta), \ F_n(t, \vartheta, L) = \sum_{p=0} (f_{np}^c \cos p\vartheta + f_{np}^s \sin p\vartheta)$$

and we should use $\vartheta_{np}(t, L)$ and $f_{np}(t, L)$ for S_0 and S_{v0}. These can then be used to construct complete functions V and F as it was done in [11]. This operation is justified when e.g. V and F are taken from the exact solution.

Let us note, that solution in the form of (4.9) apparently enables us to consider not only the simplest electrostatic beams when the emission is limited by a spatial charge, but also more complex cases when the boundary, although remaining near the axis, is connected with the fast coordinates in the manner different from (4.10) and (4.14). Here the power singularity need not be picked out in the functions V and F for the purpose of constructing the algorithm and this operation is not as critical, as in the case of flows with arbitrary cross-section which will be considered below.

Generally the deformation of coordinates as shown in (3.5), necessary in the inner domain including the boundary, becomes superfluous when the outer problem is solved. In the latter case therefore the problem reduces to changing the law of parametrization of the points of boundary $\rho = \rho_e(l)$ by means of the transformation

$$\zeta(l) = l + {}^1/_2\varepsilon^2 Q_{\rho\rho}(l)\rho_l^2(l) \qquad (4.17)$$

which introduces a correction into the right-hand side of (4.4), from $n = 6$ onwards. When a supplementary coordinate is introduced, the Cauchy conditions S_0, S_{v0} in (4.9) can be factorized in the manner analogous to (2.4). It can be shown that the second derivatives in (4.5) do not increase the order of the singularity and are compensated by a double integration in (4.9). Thus the corresponding principle of asymptotic expansion [5] is satisfied.

5. Shaping quasi-cylindrical beams of arbitrary cross section. Let

$$s = s_e(t, l), \qquad q = q_e(t, l) \tag{5.1}$$

be the parametric equations of the flow boundary weakly dependent on l. The deformed coordinate ζ is given on the surface Σ by the formulas (4.5) and (5.1). From the standpoint of the paraxial approach of Sect. 3, the inverse of this transformation is

$$l = \zeta - {}^1/_2\varepsilon^2 [Q_{ss}(\zeta) s_e^2(t, \zeta) + 2Q_{sq} s_e(t, \zeta) q_e(t, \zeta) + Q_{qq}(\zeta) q_e^2(t, \zeta)]' \tag{5.2}$$

The metric in l, s, q is given by the relation [3]

$$d\mathbf{r}^2 = [(1 - ks)^2 + \varkappa^2(s^2 + q^2)]\, dl^2 + ds^2 + dq^2 - 2\varkappa q\, dl\, ds + 2\varkappa s\, dl\, dq \tag{5.3}$$

Let us perform the conformal mapping

$$s + iq = s_e(w, \zeta) + iq_e(w, \zeta) \qquad w = u + iv \tag{5.4}$$

putting the surfaces (5.1) and the real axis $v = 0$ in the plane uv, in the 1:1 correspondence and pass to the coordinates $x^1 = \zeta$, $x^2 = u$, $x^3 = v$. Here l does not begin to deviate from ζ before the terms of the order of μ^4. Within this accuracy the (5.3) assumes the following form

$$\begin{aligned} d\mathbf{r}^2 = g_{ik}dx^i dx^k = {} & [(1 - ks)^2 + (\varkappa s + q_{,\zeta})^2 + (\varkappa q - s_{,\zeta})^2]\, d\zeta^2 + \\ & + (s_{,u}^2 + s_{,v}^2)(du^2 + dv^2) - 2[(\varkappa s + q_{,\zeta}) s_{,v} + (\varkappa q - s_{,\zeta}) s_{,u}]\, du\, d\zeta + \\ & + 2[(\varkappa s + q_{,\zeta}) s_{,u} - (\varkappa q - s_{,\zeta}) s_{,v}]\, dv\, d\zeta, \quad \sqrt{g_0} = s_{,u}^2 + s_{,v}^2, \quad s_{,u} = \partial s / \partial u \end{aligned} \tag{5.5}$$

Here k, $\varkappa$ are functions of ζ while s, q are functions of u, v, ζ, and are determined by the formula (5.4). The Laplace equation can be written in the metric (5.5), the latter enabling the correct grouping of terms of the same order of smallness. Let us separate the dependence on the deformed coordinate ζ into the fast and the slow dependence.

$$\zeta \to z, Z, \qquad \frac{\partial}{\partial \zeta} \to \frac{\partial}{\partial z} + \mu \frac{\partial}{\partial Z}, \qquad \frac{\partial^2}{\partial \zeta^2} \to \frac{\partial^2}{\partial z^2} + 2\mu \frac{\partial^2}{\partial z \partial Z} + \mu^2 \frac{\partial^2}{\partial Z^2}$$

All functions of (5.5) depend on Z, and the Cauchy conditions at the boundary Σ of the beam have in accordance with Sect. 3, the form

$$\varphi|_\Sigma = z^\nu V(t, Z, \varepsilon), \qquad \partial\varphi / \partial v|_\Sigma = z^\mu F(t, Z, \varepsilon) \tag{5.6}$$

The n-th approximation function satisfies the equation

$$\frac{\partial^2 \varphi\langle n\rangle}{\partial u^2} + \frac{\partial^2 \varphi\langle n\rangle}{\partial v^2} + \sqrt{g_0(u, v, Z)} \frac{\partial^2 \varphi\langle n\rangle}{\partial z^2} = T\langle n\rangle, \quad T\langle 0\rangle = 0 \tag{5.7}$$

For $T\langle 1\rangle$, e.g. we obtain

$$\begin{aligned} T\langle 1\rangle = -2ks\sqrt{g_0} \frac{\partial^2\varphi\langle 0\rangle}{\partial z^2} + g_{12} \frac{\partial^2\varphi\langle 0\rangle}{\partial z \partial u} + g_{13} \frac{\partial^2\varphi\langle 0\rangle}{\partial z \partial v} + \left(\frac{1}{2}\frac{\partial g_{12}}{\partial u} - \frac{\partial\sqrt{g_0}}{\partial Z}\right) \frac{\partial\varphi\langle 0\rangle}{\partial z} + \\ + ks_{,u} \frac{\partial\varphi\langle 0\rangle}{\partial u} + ks_{,v} \frac{\partial\varphi\langle 0\rangle}{\partial u} - 2\sqrt{g_0} \frac{\partial^2\varphi\langle 0\rangle}{\partial z \partial Z} \end{aligned} \tag{5.8}$$

Solution of the Cauchy problem for Eq. (5.7) with conditions (5.6) on $v = 0$ will be sought [7] using the integral representation

$$z^{\nu} \doteq \frac{1}{\Gamma(-\nu)} \int\limits_{(0)}^{\infty} \frac{e^{-pz}}{p^{\nu+1}}\, dp \tag{5.9}$$

in the following form:

$$\varphi\langle n\rangle = \int\limits_{(0)}^{\infty} \Phi_n(u, v, Z, p)\, e^{-pz} dp, \quad T\langle n\rangle = \int\limits_{(0)}^{\infty} T_n(u, v, Z, p)\, e^{-pz} dp \tag{5.10}$$

For Φ_n we obtain

$$\frac{\partial^2\Phi_n}{\partial u^2} + \frac{\partial^2\Phi_n}{\partial v^2} + \sqrt{g_0(u, v, Z)}\, p^2\Phi_n = T_n$$

$$\Phi_n\big|_{v=0} = \vartheta(u, L, p) = \frac{V(u, L, \varepsilon)}{\Gamma(-\nu)} \frac{1}{p^{\nu+1}}, \quad \frac{\partial\Phi_n}{\partial v}\Big|_{v=0} = f(u, L, p) = \frac{F(u, L, \varepsilon)}{\Gamma(-\mu)} \frac{1}{p^{\mu+1}} \tag{5.11}$$

Using the Riemann method to solve the problem (5.11) we obtain

$$\Phi_n = \mathrm{Re}\Big\{\frac{V(w, L, \varepsilon)}{\Gamma(-\nu)} \frac{1}{p^{\nu+1}} + \int\limits_0^v \Big[J_0(\lambda_e) \frac{F(\zeta)}{\Gamma(-\mu)} \frac{1}{p^{\mu+1}} -$$

$$- p^2 \frac{J_1(\lambda_e)}{\lambda_e} \{(s_e - s)\beta - (q_e - q)\alpha\} \frac{V(\zeta)}{\Gamma(-\nu)} \frac{1}{p^{\nu+1}}\Big] d\xi -$$

$$- \int\limits_0^v d\xi \int\limits_0^{v-\xi} K_n(\zeta, \eta)\, d\eta\Big\}, \quad K_n(u, v, p) = J_0(\lambda)\, T_n$$

$$\lambda_e = -pr_e = -p\,[(s_e - s)^2 + (q_e - q)^2]^{1/2}, \quad w = u + iv, \quad \zeta = u + i\xi \tag{5.12}$$

$$\lambda = -pr = -p\{[s(\zeta, \eta) - s(u, v)]^2 + [q(\zeta, \eta) - q(u, v)]^2\}^{1/2}$$

We require that the zeroth-approximation $\varphi\langle 0\rangle$ satisfies the full conditions (5.6) on the surface (5.3) and that the following approximations satisfy the homogeneous conditions on the same surface. Using the Lipschitz-Hankel integral we have

$$\varphi\langle 0\rangle = z^{\nu}\,\mathrm{Re}\, V(w, Z, \varepsilon) + \mathrm{Re}\int\limits_0^v \Big\{F(\zeta, Z, \varepsilon)(z^2 + r_e^2)^{\mu/2} P_{-\mu-1}(\Theta_e) -$$

$$- \frac{(s_e - s)\beta - (q_e - q)\alpha}{(z^2 + r_e^2)^{1-\nu/2}} \frac{dP_{-\nu}}{d\Theta_e} V(\zeta, Z, \varepsilon)\Big\} d\xi, \quad \Theta_e = \frac{z}{\sqrt{z^2 + r_e^2}}, \quad \zeta = u + i\xi \tag{5.13}$$

When the emission is limited by a spatial charge, we have $\nu = \mu = 4/3$. The n-th approximation function $\varphi\langle n\rangle$ is given by the formula

$$\varphi\langle n\rangle = -\int\limits_{(0)}^{\infty} e^{-pz} dp \int\limits_0^v d\xi \int\limits_0^{v-\xi} K_n(\zeta, \eta, p)\, d\eta \tag{5.14}$$

The contour integral in (5.14), has unfortunately no closed expression. The approach adopted in [12] in constructing the shaping electrodes for a conical beam should be used to obtain its estimate. We note that since

$$dJ_0 / dx = -J_1, \qquad dJ_1 / dx = -J_1 / x + J_0$$

all approximations will contain only J_0 and J_1. Thus, in the first approximation, it is necessary to know the integrals containing the products $J_0(ap)\,J_0(bp)$, $J_0(ap)\,J_1(bp)$, while in the second approximation the integral includes various cubic combinations of the Bessel functions.

5.1. Toroidal beam of arbitrary cross section. Let us select a beam of required configuration out of a flow defined by an exact solution [8]. This beam starts from a flat emitter in such a way that the l, s, q coordinate system with a circle as the axial curve ($k = 1/R_0 =$const, $\varkappa = 0$) is found to be orthogonal and related to the emitter and the trajectories; in this case the coordinates need not be deformed. The surface of the beam cylindrical in l, s, q

$$s = s_e(t),\ q = q_e(t);\ s + iq = s_e(w) + iq_e(w),\quad w = u + iv \tag{5.15}$$

In the l, u, v coordinate system the metric is defined by the formulas

$$g_{11} = (1 - ks)^2,\ g_{22} = g_{33} = s_{,u}{}^2 + s_{,v}{}^2,\ \sqrt{g} = (1 - ks)(s_{,u}{}^2 + s_{,v}{}^2)$$

We now assume with respect to the contour (5.15) that the quasi-cylindrical angle ϑ can be used to obtain a single-valued parametric representation. The potential and the normal derivative on $v = 0$ will then be given by

$$\varphi|_{v=0} = V(z, L, \vartheta) = \left(\frac{3}{2}z\right)^{4/3} V(L,\vartheta);\quad V(L,\vartheta) = \frac{1}{[1 - ks_e(\vartheta)]^{10/3}}\left(\frac{\sin x}{x}\right)^{4/3}$$

$$\frac{\partial\varphi}{\partial v}\bigg|_{v=0} = F(z, L, \vartheta) = \left(\frac{3}{2}z\right)^{4/3} F(L,\vartheta),\ F(L,\vartheta) = -\frac{2k\beta(\vartheta)}{[1 - ks_e(\vartheta)]^{13/3}}\left(\frac{\sin x}{x}\right)^{1/3}\left(\frac{\sin x}{x} + \cos x\right) \tag{5.16}$$

$$x = {}^3/{}_2 kL\,[1 - ks_e(\vartheta)]^{-1},\qquad \beta(\vartheta) = dq_e/d\vartheta$$

In this case the right-hand sides of (5.7) and (5.11) can be written for any value of n as

$$T_n = k\left[2s\left(\frac{\partial^2\Phi_{n-1}}{\partial u^2} + \frac{\partial^2\Phi_{n-1}}{\partial v^2}\right) + s_{,u}\frac{\partial\Phi_{n-1}}{\partial u} + s_{,v}\frac{\partial\Phi_{n-1}}{\partial v}\right] +$$
$$+ 2(s_{,u}{}^2 + s_{,v}{}^2)\,p\,\frac{\partial\Phi_{n-1}}{\partial L} - k^2\left[s^2\left(\frac{\partial^2\Phi_{n-2}}{\partial u^2} + \frac{\partial^2\Phi_{n-2}}{\partial v^2}\right) +\right.$$
$$\left. + ss_{,u}\frac{\partial\Phi_{n-2}}{\partial u} + ss_{,v}\frac{\partial\Phi_{n-2}}{\partial v}\right] - (s_{,u}{}^2 + s_{,v}{}^2)\frac{\partial^2\Phi_{n-2}}{\partial L^2} \tag{5.17}$$

We note that the second derivatives with respect to the variables u, v appearing in T_n can be completely eliminated.

As we noted before, the asymptotic expansions quoted are applicable throughout the strip of width μ. However, when the geometry is sufficiently complex, the use of these expressions away from the emitter may be found inexpedient when we consider the volume of the necessary computations. They can however be replaced by quasi-onedimensional expansions [3] or directly with series written in terms of the coordinate normal to the boundary of the flow [6].

Use of the Cauchy conditions (Sect. 3) in the expressions given in Sect. 4 and 5 provides a complete solution of the inverse problem of the theory of intense beams in the paraxial approximation.

BIBLIOGRAPHY

1. Ovcharov, V. T., Equations of the electronic optics for the plane-symmetric and axisymmetric beams with high current density. Radiotekhnika and electronika, Vol. 7 No. 8, 1962.
2. Kirstein, P. T., Paraxial formulation of the equations of electrostatic space-charge flow. J. Appl. Phys., Vol. 30, No. 7, 1959.
3. Danilov, V. N., Paraxial approximation for a dense electronic beam. PMTF, No. 5, 1968.
4. Friedrichs, K. O., Asymptotic phenomena in mathematical physics. Bull. Amer. Math. Soc., Vol. 61, No. 6, 1955.
5. Van Dyke, Milton, Perturbation Methods in Fluid Mechanics. Academic Press, N.Y. and London, 1964.
6. Syrovoi, V. A., On the theory of electrostatic focussing of intense beams of charged particles. PMTF, No. 4, 1967.
7. Syrovoi, V. A., Solving the Cauchy problem for the Laplace equation in the three-dimensional case with special reference to the problem of shaping of intense charged-particle beams. PMM Vol. 34, No. 1, 1970.
8. Meltzer, B., Single component stationary electron flow under space-charge conditions. J. Electronics, Vol. 2, No. 2, 1956.
9. Radley, D. E., The theory of the Pierce-type electron guns. J. Electr. Contr., Vol. 4, No. 2, 1958.
10. Lomax, R. J., Exact electrode systems for the formation of a curved space-charge beam. J. Electr. Contr., Vol. 3, No. 4, 1957.
11. Kuznetsov, Iu. E. and Syrovoi, V. A., On solution of equations of regular electrostatic beam emitted from an arbitrary surface. PMTF, No. 2, 1966.
12. Radley, D. E., Electrodes for convergent Pierce-type electron guns. J. Electr. Contr., Vol. 15, No. 5, 1963.

PRESSURE OF A PLANE CIRCULAR STAMP ON AN ELASTIC HALF-SPACE WITH AN INDENTATION OR INCLUSION

PMM Vol. 35, No. 4, 1971, pp. 669-676

V. G. BOGOVOI and B. M. NULLER
(Leningrad)

(Received December 29, 1970)

An axisymmetric mixed problem of the theory of elasticity for a half-space with a hemispherical indentation of radius $\rho < 1$ is considered. The boundary of the half-space is acted upon by a plane circular stamp of unit radius, coaxial with the indentation and covering it completely. There is no friction between the stamp and the half-space. The problem is solved for three cases: the indentation may be empty, or filled with either a perfectly rigid, or a perfectly elastic medium. The solution is constructed in the form of series in terms of the homogeneous solutions of the mixed problem for a half-space, and the